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Algebraic entropy and the space of initial values for discrete dynamical systems

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Abstract

A method to calculate the algebraic entropy of a mapping, which can be lifted to an isomorphism of a suitable rational surface (the space of initial values), is presented. It is shown that the degree of the nth iterate of such a mapping is given by its action on the Picard group of the space of initial values. It is also shown by construction that the degree of the nth iterate of every Painlevé equation in Sakai's list is $O(n^2)$ and therefore its algebraic entropy is zero.

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1. Introduction

The notion of algebraic entropy was introduced by Hietarinta and Viallet [1] in order to test the degree of complexity of successive iterations of a rational mapping. The algebraic entropy is defined as $s := \lim_{n\to\infty} \log(d_n)/n$ where d_n is the degree of the nth iterate. This notion is linked to Arnold's complexity since the degree of a mapping gives the intersection number of the image of a line and a hyperplane. While the degree grows exponentially for a generic mapping, it was shown that it only grows polynomially for a large class of integrable mappings [1–3]. In particular, the case of some discrete Painlevè equations is studied by Ohta *et al* [4].

Let φ_i be a birational mapping of $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. A sequence of rational surfaces X_i is (or X_i themselves are) called the space of initial values for the sequence of φ_i if each φ_i is lifted to an isomorphism, i.e. bi-holomorphic mapping, from X_i to X_{i+1} [5–7]. Here, the mapping φ' is called a mapping lifted from the mapping φ if φ' coincides with φ at any point where φ is defined. Such a mapping induces an action on the Picard group of its space of initial values. Here, the Picard group of a rational surface X is the group of isomorphism classes of invertible sheaves on X and it is isomorphic to the group of linear equivalence classes of divisors on X.

In this paper we present some basic formulae to calculate the degree of the nth iterate of the sequence of rational mappings, which is proposed in the previous paper by the author [6] for birational mappings with the space of initial values. In the case where the mappings are birational and have the space of initial values the calculation reduces to the calculation of the power of some matrix. This method is essentially the projection of the formula which has been proposed [8, 9] for automorphisms of rational surfaces to the formula for birational mappings on a minimal surface (or $\mathbb{C}^2$). We also show examples of calculation and simplify the method by considering invariant sublattices. We also apply our method to the discrete Painlevé equations in Sakai's list [10] and prove that for all of them the degrees grow at most in the order n^2. This proof is based on their construction and the corresponding root systems.

The discrete Painlevé equations have been found by many authors [11, 12] and have been extensively studied. Recently it was shown by Sakai [10] that all (from the point of view of symmetries) of these are obtained by studying rational surfaces in connection with the extended affine Weyl groups.

Surfaces obtained by successive blow-ups of $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ have been studied by several authors by means of connections between the Weyl groups and the groups of Cremona isometries on the Picard group of the surfaces [13–15]. Here, a Cremona isometry is an isomorphism of the Picard group such that (a) it preserves the intersection number of any pair of divisors, (b) it preserves the canonical divisor K_X and (c) it leaves the set of effective classes of divisors invariant. In the case where nine points (in the case of $\mathbb{P}^2$, eight points in the case of $\mathbb{P}^1 \times \mathbb{P}^1$) are blown up, if the points are in a general position the group of Cremona isometries becomes isomorphic with an extension of the Weyl group of type $E_8^{(1)}$. When the nine points are not in a general position, the classification of connections between the groups of Cremona isometries and the extended affine Weyl groups was first studied by Looijenga [16] and more generally by Sakai. Birational (bi-meromorphic) mappings on $\mathbb{P}^2$ (or $\mathbb{P}^1 \times \mathbb{P}^1$) are obtained by interchanging the procedure of blow-downs. Discrete Painlevé equations are recovered as the birational mappings corresponding to the translations of the affine Weyl groups.

In section 2, we show a method to calculate the degree of the nth iterate. Considering the intersection numbers of divisors it is shown that for general rational mappings the degree is given by the action of the mapping on the Picard group. In the case where the mapping is birational and has the space of initial values, i.e. it is lifted to isomorphism between certain surfaces, it is given by the nth power of the corresponding matrix.

In section 3, we present an example of calculation. We apply our method to the mapping which was found by Hietarinta and Viallet [1] and whose space of initial values is obtained by 14 blow-ups from $\mathbb{P}^1 \times \mathbb{P}^1$ [6]. We simplify the calculation using the root systems associated with the symmetries of their space of initial values.

In section 4, applying our method for discrete Painlevé equations, we show by using the construction that the degrees of the nth iterate are $O(n^2)$.

2. Algebraic entropy and intersection numbers

This section is devoted to discussing the relation between the degree of a (bi-)rational mapping and the corresponding surfaces by using the basic theory of algebraic geometry. We show a method to calculate the degree of the nth iterate (where n is finite) by using the theory of intersection numbers for general (bi-)rational mappings. This method is essentially based on the elimination of indeterminacy and the projection formula. Notice that we have to calculate successive elimination to obtain the sequence of surfaces in this method and therefore this method does not give a formula for the degree of the mapping as $n \to \infty$ immediately. Next we adapt this method to birational mappings which have the space of initial values and show that the degree of mapping is given by the nth power of a matrix.

Let φ_i: $(x, y) \in \mathbb{C}^2 \mapsto (\overline{x}, \overline{y}) \in \mathbb{C}^2$ be a rational mapping for each $i = 0, 1, 2, \ldots$. We can relate a mapping φ_i': $(X : Y : Z) \in \mathbb{P}^2 \mapsto (\overline{X} : \overline{Y} : \overline{Z}) = (f_i(X, Y, Z) : g_i(X, Y, Z) : h_i(X, Y, Z)) \in \mathbb{P}^2$ to the mapping φ_i by using the relations $x = X/Z$, $y = Y/Z$

and $\overline{x} = \overline{X}/\overline{Z}, \overline{y} = \overline{Y}/\overline{Z}$ and by reducing to a common denominator. We denote $\varphi'^n := \varphi'_{n-1} \circ \cdots \circ \varphi'_1 \circ \varphi'_0$ by $(X_n : Y_n : Z_n) = (f^n(X_0, Y_0, Z_0) : g^n(X_0, Y_0, Z_0) : h^n(X_0, Y_0, Z_0))$ where f^n, g^n, h^n are polynomials with the same degree and should be simplified if possible. It is easily shown that $\varphi'_{n-1} \circ \cdots \circ \varphi'_1 \circ \varphi'_0 = (\varphi_{n-1} \circ \cdots \circ \varphi_1 \circ \varphi_0)'$. We denote the coordinates X, Y, Z by x, y, z and φ' by φ for simplicity and avoiding confusion of notations. The degree of the sequence of mappings is defined by the degree of polynomials and the algebraic entropy h of $\{\varphi_i\}$ is defined by

$$h = \lim_{n\to\infty} \frac{1}{n} \log \deg(\varphi^n)$$

if the limit exists.

Let $\{\varphi_i\}$ be a sequence of rational mappings on $\mathbb{P}^2$. First, blowing up $\mathbb{P}^2$, we can construct the surfaces $Y_{1,i}$ such that each mapping φ_i can be lifted to a regular mapping from $Y_{1,i}$ to $Y_{0,i+1} := \mathbb{P}^2$, where the mapping ψ' is called a mapping lifted from the mapping ψ if ψ' coincides with ψ at any point where ψ is defined. In our case φ_i can also be lifted to a rational mapping from $Y_{1,i}$ to $Y_{1,i+1}$ and hence similarly φ_i is lifted to a rational mapping from $Y_{2,i}$ to $Y_{2,i+1}$. Continuing this operation we obtain surfaces $Y_{k,i}$ for $k, i = 0, 1, 2, \ldots$ such that φ_i is lifted to a rational mapping from $Y_{k,i}$ to $Y_{k,i+1}$ and is also lifted to a regular mapping from $Y_{k+1,i}$ to $Y_{k,i+1}$.

Let a, b, c be complex numbers. The total transform of the line $ax_{i+1} + by_{i+1} + cz_{i+1} = 0$ in $Y_{k,i+1}$ coincides with the proper transform of the line for generic a, b, c and we denote it by L. We denote the linear equivalence class of the divisor of a line in $\mathbb{P}^2$ by $\mathcal{E}$. The pre-image of L by $\varphi^i : Y_{k+1,i} \to Y_{k,i+1}$ is the total transform of

$$af_i(x_i, y_i, z_i) + bg_i(x_i, y_i, z_i) + ch_i(x_i, y_i, z_i) = 0$$

in $Y_{k+1,i}$. We denote the action of the pull-back of φ_i from the Picard group of $Y_{k,i+1}$ to that of $Y_{k+1,i}$ by φ_i^*.

By Bézout's theorem the degree of f_i coincides with the intersection number of $\varphi_i^*(\mathcal{E})$ and a generic line in $\mathbb{P}^2$. Notice that the intersection points of $\varphi_i^*(\mathcal{E})$ and a generic line are not blown up. Hence we have the formula

$$\deg(\varphi_i) = \varphi_i^*(\mathcal{E}) \cdot \mathcal{E} \tag{1}$$

where $\cdot$ denotes the intersection number of a pair of divisors.

Similarly for the mapping $\varphi^n : Y_{n+k,0} \to Y_{k,n}$ we have the formula

$$\deg(\varphi^n) = (\varphi^n)^*(\mathcal{E}) \cdot \mathcal{E} \tag{2}$$

where $(\varphi^n)^*$ denotes the action of the pull-back of φ^n from the Picard group of $Y_{k,n}$ to that of $Y_{k+n,0}$.

Remark. Let X be a surface obtained by blowing up $\mathbb{P}^2$ L times. We write the Picard group of X as $\mathrm{Pic}(X)$. The Picard group $\mathrm{Pic}(X)$ is a $\mathbb{Z}$-module in the form

$$\mathbb{Z}\mathcal{E} + \sum_{l=1}^{L} \mathbb{Z}E_l$$

where E_l denotes the linear equivalence class of the total transform of the point of the lth blow-up. The action $(\varphi_i)^*$ is a linear transformation of the lattices. The intersection numbers are given by the intersection form

$$\mathcal{E} \cdot \mathcal{E} = 1 \qquad \mathcal{E} \cdot E_l = 0 \qquad E_l \cdot E_m = \delta_{l,m} \tag{3}$$

where $\delta_{i,j}$ is 1 if $i = j$ and 0 if $i \neq j$, and the intersection numbers of any pairs of divisors are given by their linear combinations. Hence the calculation of the degree reduces to a calculation of linear algebra in principle. The remaining difficulty is the calculation of the sequence of surfaces as $n \to \infty$.

Next we present another calculation for general birational mappings by using the inverse mappings. Let $\{\varphi_i\}$ be a sequence of birational mappings. Similar to the forward mappings we can construct surfaces $Z_{k,i}$ for $k, i = 0, 1, 2, \ldots$ such that φ_i^{-1} is lifted to a birational mapping from $Z_{k,i+1}$ to $Z_{k,i}$ and is also lifted to a birational regular mapping from $Z_{k+1,i+1}$ to $Z_{k,i}$.

Let a, b, c be complex numbers. The total transform of the line $ax_i + by_i + cz_i = 0$ in $Z_{k,i}$ coincides with the proper transform of the line for generic a, b, c and we denote it by L. The line L is written by the parameter $s : t \in \mathbb{P}^1$ as $(x_i : y_i : z_i) = (cs : ct : -as - bt)$. The pre-image of L by $\varphi_i^{-1} : Z_{k+1,i+1} \to Z_{k,i}$ is the total transform of the curve $(f_i(cs : ct : -as - bt), g_i, h_i)$ in $Z_{k+1,i+1}$. We denote the action of the pull-back of φ_i^{-1} from the Picard group of $Z_{k,i}$ to that of $Z_{k+1,i+1}$ by $(\varphi_i^{-1})^*$.

The degree of $\varphi^n((cs : ct : -as - bt))$ as polynomials of $s : t$ coincides with the degree of $(f_i(cs : ct : -as - bt), g_i, h_i)$. Moreover since the intersection number of $(\varphi_i^{-1})^*(\mathcal{E})$ and a generic line in $\mathbb{P}^2$ is given by the equation

$$\text{(a linear combination of } f_i(cs : ct : -as - bt), g_i \text{ and } h_i) = 0$$

in $\{(s : t) \in \mathbb{P}^1\}$, it coincides with the degree of φ_i. Hence we have the formula

$$\deg(\varphi_i^{-1}) = (\varphi_i^{-1})^*(\mathcal{E}) \cdot \mathcal{E}. \tag{4}$$

Similarly for the mapping $\varphi^{-n} := (\varphi^n)^{-1} : Z_{n+k,n} \to Z_{k,0}$, we have the formula

$$\deg(\varphi^n) = (\varphi^{-n})^*(\mathcal{E}) \cdot \mathcal{E} \tag{5}$$

where $(\varphi^{-n})^*$ denotes the action of the pull-back of φ^{-n} from the Picard group of $Z_{k,0}$ to that of $Z_{k+n,n}$.

As a corollary of the formulae (1) and (5) (or (2) and (4)) we have

$$\deg(\varphi^n) = \deg(\varphi^{-n}). \tag{6}$$

Next, we consider the case where the mappings φ_i have the space of initial values. Let $\{X_i\}$ be a sequence of surfaces obtained by blowing up $\mathbb{P}^2$ such that each φ_i is lifted to an isomorphism from X_i to X_{i+1}. Similar to the birational case, we have the formula

$$\begin{aligned} \deg(\varphi_i) &= \varphi_i^*(\mathcal{E}) \cdot \mathcal{E} = (\varphi_i^{-1})^*(\mathcal{E}) \cdot \mathcal{E} \\ \deg(\varphi^n) &= (\varphi^n)^*(\mathcal{E}) \cdot \mathcal{E} = (\varphi^{-n})^*(\mathcal{E}) \cdot \mathcal{E} \end{aligned} \tag{7}$$

where $(\varphi_i)^*$ denotes the action from $\mathrm{Pic}(X_{i+1})$ to $\mathrm{Pic}(X_i)$ and $(\varphi_i^{-1})^*$ denotes the action from $\mathrm{Pic}(X_i)$ to $\mathrm{Pic}(X_{i+1})$ and so on. Here the Picard group of X_i is isomorphic to each other, hence the action of $(\varphi^n)^*$ and $(\varphi^{-n})^*$ are linear isomorphisms on $\mathbb{Z}^{K+1}$, where K is the number of blow-ups. In the case where each φ_i^* is the same transformation with respect to i, as in the case of the discrete Painlevé equations, $(\varphi^n)^*$ $(= (\varphi_0^n)^*)$ is the power of the corresponding matrix, so we can observe the behaviour of the degree of φ^n as $n \to \infty$.

Consequently we have the following results.

Proposition 2.1.

(i) Let each φ_i be a rational mapping on $\mathbb{P}^2$; the formula

$$\deg(\varphi^n) = (\varphi^n)^*(\mathcal{E}) \cdot \mathcal{E} \tag{8}$$

holds, where φ^n is considered to be a regular mapping from $Y_{n+k,0}$ to $Y_{k,n}$.

(ii) Let each φ_i be a birational mapping on $\mathbb{P}^2$; the formula

$$\deg(\varphi^n) = \deg(\varphi^{-n}) = (\varphi^n)^*(\mathcal{E}) \cdot \mathcal{E} = (\varphi^{-n})^*(\mathcal{E}) \cdot \mathcal{E} \tag{9}$$

holds, where φ^n and φ^{-n} are considered to be a birational regular mapping from $Y_{n+k,0}$ to $Y_{k,n}$ and that from $Z_{n+k,n}$ to $Z_{k,0}$, respectively.

(iii) Let each φ_i be a birational mapping such that they have the space of initial values; the formula (9) also holds, where φ^n and φ^{-n} are considered to be an isomorphism from X_0 to X_n and one from X_n to X_0 respectively. Moreover in the case where $\varphi^* : \mathrm{Pic}(X_{i+1}) \to \mathrm{Pic}(X_i)$ is the same transformation with respect to i, each $(\varphi^n)^*$ $(= (\varphi_0^*)^n)$ and $(\varphi^{-n})^*$ $(= ((\varphi_0^{-1})^*)^n)$ is the power of the corresponding matrix respectively.

Next, for the sake of convenience, we present the corresponding formulae in the case where φ_i is considered to be a rational mapping on $\mathbb{P}^1 \times \mathbb{P}^1$. (In this paper the definition of degree is slightly modified from [6].)

We denote the degree of a polynomial on $\mathbb{C}$: $f(t) = \sum_m a_t t^m$ by $\deg f(t)$ $(= \deg_t f(t))$. The degree of a rational function on $\mathbb{P}^1$, which is written as $P(x) = f(x)/g(x)$ on one of the local coordinates, where $f(x)$ and $g(x)$ are polynomials, is defined by

$$\deg(P) = \max\{\deg f(x), \deg g(x)\}.$$

The degree of a rational function on $\mathbb{P}^1 \times \mathbb{P}^1$, which is written as $P(x, y) = f(x, y)/g(x, y)$ on one of the local coordinates, where $f(x, y)$ and $g(x, y)$ are polynomials, is defined by

$$\deg(P) = \deg_x P(x, y) + \deg_y P(x, y).$$

The degree of a mapping $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, $(x, y) \mapsto (P(x, y), Q(x, y))$, where $P(x, y)$ and $Q(x, y)$ are rational functions, is defined by

$$\deg(\varphi) = \max\{\deg P(x, y), \deg Q(x, y)\}$$

and similarly $\deg_t(\varphi)$ is defined by the degree about t.

Let each $\varphi_i : (x, y) \mapsto (\overline{x}, \overline{y}) = (P_i(x, y), Q_i(x, y))$, where P_i, Q_i are rational functions, be a rational mapping on $\mathbb{P}^1 \times \mathbb{P}^1$. We denote $\varphi^n := \varphi_{n-1} \circ \cdots \circ \varphi_1 \circ \varphi_0$ by $(x_n, y_n) = (P^n(x_0, y_0), Q^n(x_0, y_0))$. The algebraic entropy h of $\{\varphi_i\}$ is defined by

$$h = \lim_{n\to\infty} \frac{1}{n} \log \deg(\varphi^n)$$

if the limit exists. It is easily shown that this algebraic entropy coincides with that in the $\mathbb{P}^2$ case.

Let $\{\varphi_i\}$ be a sequence of rational mappings on $\mathbb{P}^1 \times \mathbb{P}^1$. Let $Y_{k,i}$ be obtained by blowing up such that φ_i is lifted to a rational mapping from $Y_{k,i}$ to $Y_{k,i+1}$ and is also lifted to a regular mapping from $Y_{k+1,i}$ to $Y_{k,i+1}$ as in the $\mathbb{P}^2$ case. The total transform of the line $x_{i+1} = d$ in $Y_{k,i+1}$ coincides with the proper transform of the line for generic d and we denote it by L_x. We denote the linear equivalence class of the divisor of $x = d$ in $\mathbb{P}^1 \times \mathbb{P}^1$ by H_0 and that of $y = d'$ by H_1.

The pre-image of L_x by $\varphi^i : Y_{k+1,i} \to Y_{k,i+1}$ is the total transform of

$$P_i(x_i, y_i) = d$$

in $Y_{k+1,i}$. Since the intersection number of $\varphi_i^*(H_0)$ and the line L_y: $y_i = e$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is given by the number of solutions of the equation

$$P_i(x_i, e) = d$$

it coincides with $\deg_{x_i} P_i$. Hence we have the formula

$$\deg_{x_i}(P_i) = (\varphi_i)^*(H_0) \cdot H_1$$

where we use the fact that the linear equivalence class of the divisor of L_y is H_1 and the fact that the intersection points are not blown up for generic L_y. Analogously, we have the formula

$$\deg_{y_i}(P_i) = (\varphi_i)^*(H_0) \cdot H_0$$
$$\deg_{x_i}(Q_i) = (\varphi_i)^*(H_1) \cdot H_1$$
$$\deg_{y_i}(Q_i) = (\varphi_i)^*(H_1) \cdot H_0$$

and the formulae

$$\deg_x P^n(x, y) = (\varphi^n)^*(H_0) \cdot H_1$$
$$\deg_y P^n(x, y) = (\varphi^n)^*(H_0) \cdot H_0$$
$$\deg_x Q^n(x, y) = (\varphi^n)^*(H_1) \cdot H_1$$
$$\deg_y Q^n(x, y) = (\varphi^n)^*(H_1) \cdot H_0.$$

Remark. Let X be a surface obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ L times. The Picard group $\mathrm{Pic}(X)$ is a $\mathbb{Z}$-module in the form

$$\mathbb{Z}H_0 + \mathbb{Z}H_1 + \sum_{l=1}^{L} \mathbb{Z}E_l.$$

The intersection form is

$$H_i \cdot H_j = 1 - \delta_{i,j} \qquad E_l \cdot E_m = -\delta_{l,m} \qquad H_i \cdot E_l = 0.$$

Next we consider the case where each φ_i is a birational mapping on $\mathbb{P}^1 \times \mathbb{P}^1$. Let $Z_{j,i}$ be obtained by blowing up such that φ_i^{-1} is lifted to a birational mapping from $Z_{k,i+1}$ to $Z_{k,i}$ and is also lifted to a birational regular mapping from $Z_{k+1,i+1}$ to $Z_{k,i}$ as in the $\mathbb{P}^2$ case. We denote the line $y_i = e$ in $Z_{k,i}$ by L_y. The line L_y is written by the parameter $t \in \mathbb{C}$ as $(x_i, y_i) = (t, e)$. The pre-image of L_y by $\varphi_i^{-1} : Z_{k+1,i+1} \to Z_{k,i}$ is the total transform of the curve $(P_i(t, e), Q_i(t, e))$ in $Z_{k+1,i+1}$ and $\deg_t P_i(t, e)$ coincides with $\deg_{x_i} P_i$. Moreover $\deg_t P_i(t, e)$ coincides with the intersection number of $(\varphi_i^{-1})^*(L_y)$ and a line $L_x : x_{i+1} = d$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Hence we have the formulae

$$\deg_{x_i}(P_i) = (\varphi_i^{-1})^*(H_1) \cdot H_0$$
$$\deg_{y_i}(P_i) = (\varphi_i^{-1})^*(H_0) \cdot H_0$$
$$\deg_{x_i}(Q_i) = (\varphi_i^{-1})^*(H_1) \cdot H_1$$
$$\deg_{y_i}(Q_i) = (\varphi_i^{-1})^*(H_0) \cdot H_1$$

and the formulae

$$\deg_x P^n(x, y) = (\varphi^{-n})^*(H_1) \cdot H_0$$
$$\deg_y P^n(x, y) = (\varphi^{-n})^*(H_0) \cdot H_0$$
$$\deg_x Q^n(x, y) = (\varphi^{-n})^*(H_1) \cdot H_1$$
$$\deg_y Q^n(x, y) = (\varphi^{-n})^*(H_0) \cdot H_1.$$

In the case where the φ_i have the space of initial values, the statement corresponding to (iii) in proposition 2.1 also holds.

Consequently we have the following formulae:

$$\deg P^n(x, y) = (\varphi^n)^*(H_0) \cdot (H_0 + H_1) = (\varphi^{-n})^*(H_0 + H_1) \cdot H_0$$
$$\deg Q^n(x, y) = (\varphi^n)^*(H_1) \cdot (H_0 + H_1) = (\varphi^{-n})^*(H_0 + H_1) \cdot H_1$$

where the left-hand equalities hold for general rational mappings and the right-hand equalities hold for birational mappings.

3. Examples and simplification

In this section we present two examples. The first one is the birational mapping found by Hietarinta and Viallet [1] (we denote it as the HV equation in this paper). It is known that its algebraic entropy is positive and that it is associated with a root system of indefinite type. We show that the algebraic entropy can be computed by using the action on the root lattice. The second example is the second discrete Painlevé equation, $\mathrm{dP_{II}}$. We present that the order of its nth iterate is n^2. It is known that $\mathrm{dP_{II}}$ is associated with the root system of affine type ($A_2^{(1)}$), but we see that the algebraic entropy cannot be computed only by the action on the root lattice in this case.

3.1. Hietarinta–Viallet equation

We consider the HV equation written as follows:

$$\begin{aligned}&\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1\\ &\begin{pmatrix} x_n \\ y_n \end{pmatrix} \mapsto \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} y_n \\ -x_n + y_n + a/y_n^2 \end{pmatrix}\end{aligned} \tag{10}$$

where $a \in \mathbb{C}$ is a nonzero constant. It is known that the algebraic entropy of the HV equation φ is equal to $\log(3+\sqrt{5})/2$. Here we shall recover the algebraic entropy of the HV equation by using the theory of intersection numbers.

By the change of variables $x = X/Z$, $y = Y/Z$, this mapping reduces to the mapping $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$

$$\varphi : (X, Y, Z) \mapsto (Y^3, -XY^2 + Y^3 + aZ^3, Y^2 Z).$$

The HV equation can be lifted to an automorphism of a rational surface X obtained by 15 successive blow-ups from $\mathbb{P}^2$. Hence its Picard group is

$$\mathrm{Pic}(X) = \mathbb{Z}\mathcal{E} + \mathbb{Z}E_1 + \mathbb{Z}E_2 + \cdots + \mathbb{Z}E_{14}$$

where total transforms of the points of blow-ups are as follows (figure 1):

$$\begin{aligned}
&E_1 : (Y/X, Z/X) = (0, 0)\\
&E_2 : (u_1, v_1) := (Z/X, Y/Z) = (0, 0)\\
&E_3 : (u_2, v_2) := (u_1/v_1, v_1) = (0, 0)\\
&E_4 : (u_3, v_3) := (u_2, v_2/u_2) = (0, a)\\
&E_5 : (u_4, v_4) := (u_3, (v_3 - a)/u_3) = (0, 0)\\
&E_6 : (Z/Y, X/Y) = (0, 0)\\
&E_7 : (u_6, v_6) := (Z/Y, X/Z) = (0, 0)\\
&E_8 : (u_7, v_7) := (u_6/v_6, u_6) = (0, 0)\\
&E_9 : (u_8, v_8) := (u_7, v_7/v_7) = (0, a)\\
&E_{10} : (u_9, v_9) := (u_8, (v_8 - a)/u_8) = (0, 0)\\
&E_{11} : (Z/X, Y/X) = (0, 1)\\
&E_{12} : (u_{11}, v_{11}) := (Z/X, (Y - X)/Z) = (0, 0)\\
&E_{13} : (u_{12}, v_{12}) := (u_{11}/v_{11}, v_{11}) = (0, 0)\\
&E_{14} : (u_{13}, v_{13}) := (u_{12}, v_{12}/u_{12}) = (0, a)\\
&E_{15} : (u_{14}, v_{14}) := (u_{13}, (v_{13} - a)/u_{13}) = (0, 0).
\end{aligned}$$

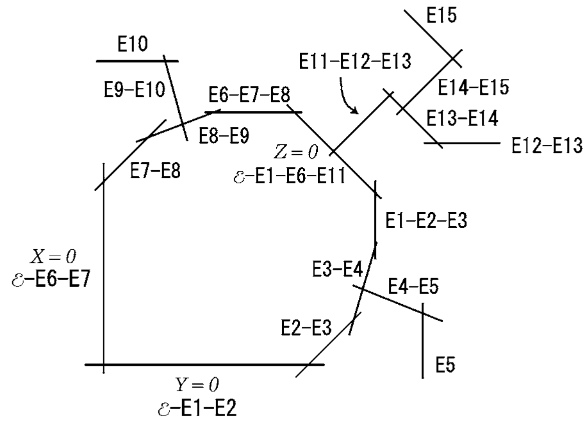

Figure 1. Proper transformation of the HV equation.

The action of φ $(= (\varphi^{-1})^*)$ on the Picard group is

$$\varphi : \begin{pmatrix} \mathcal{E} \\ E_1 \\ E_2, \ E_3 \\ E_4, \ E_5 \\ E_6, \ E_7, \ E_8, \ E_9, \ E_{10} \\ E_{11}, \ E_{12}, \ E_{13}, \ E_{14}, \ E_{15} \end{pmatrix} \mapsto \begin{pmatrix} 3\mathcal{E} - 2E_6 - E_7 - E_8 - E_9 - E_{10} \\ 2\mathcal{E} - E_6 - E_7 - E_8 - E_9 - E_{10} \\ \mathcal{E} - E_6 - E_{10}, \ \mathcal{E} - E_6 - E_9 \\ \mathcal{E} - E_6 - E_8, \ \mathcal{E} - E_6 - E_7 \\ E_{11}, \ E_{12}, \ E_{13}, \ E_{14}, \ E_{15} \\ E_1, \ E_2, \ E_3, \ E_4, \ E_5 \end{pmatrix} \tag{11}$$

(this table means $\overline{\mathcal{E}} = 3\mathcal{E} - 2E_6 - E_7 - E_8 - E_9 - E_{10}$, $\overline{E_1} = 2\mathcal{E} - E_6 - E_7 - E_8 - E_9 - E_{10}$, $\overline{E_2} = \mathcal{E} - E_6 - E_{10}$ and so on) and their linear combinations.

Notice that (11) means a change of bases. Actually by fixing the basis of Pic(X) as $\{\mathcal{E}, E_1, E_2, \ldots, E_{15}\}$, this action can be expressed by the corresponding matrix as the action from the left-hand side on the space of coefficients of basis.

The action of φ on Pic(X) is given by (11). Hence the algebraic entropy of the HV equation, $\lim_{n\to\infty} \frac{1}{n} \log \varphi^n(\mathcal{E}) \cdot \mathcal{E}$, can be shown to be equal to (by diagonalization of the corresponding matrix of (11))

$$\log \max\{|\text{ eigenvalues of } (11)|\} = \log \frac{3+\sqrt{5}}{2}.$$

On the level of the mapping itself, the degrees can be calculated as follows:

$$\begin{aligned} (X, Y, Z) &\xrightarrow{\varphi'} (Y^3, -XY^2 + Y^3 + aZ^3, Y^2 Z) \xrightarrow{\varphi'} \deg 9 \\ &\xrightarrow{\varphi'} \deg 27 \xrightarrow{\varphi'} \deg 73 \xrightarrow{\varphi'} \cdots. \end{aligned} \tag{12}$$

On the other hand, the intersection numbers can be calculated by (11) as follows:

$$\begin{aligned} \mathcal{E} &\xrightarrow{\varphi} 3\mathcal{E} - 2E_6 - E_7 - E_8 - E_9 - E_{10} \\ &\xrightarrow{\varphi} 9\mathcal{E} + \cdots \xrightarrow{\varphi} 27\mathcal{E} + \cdots \xrightarrow{\varphi} 73\mathcal{E} + \cdots \\ &\xrightarrow{\varphi} \cdots \end{aligned}$$

and hence $\varphi^n(\mathcal{E}) \cdot \mathcal{E}$ actually coincides with (12).

Next we consider simplification of our method. The anti-canonical divisor $-K_X$ can be reduced uniquely [7] to prime divisors as

$$D_0 + 2D_1 + D_2 + D_3 + 2D_4 + D_5 + 3D_6 + D_7 + 2D_8 + D_9 + 2D_{10} + 2D_{11} + 2D_{12}$$

where

$D_0 = E_2 - E_3 \quad D_1 = E_3 - E_4 \quad D_2 = E_4 - E_5 \quad D_3 = E_7 - E_8 \quad D_4 = E_8 - E_9$

$D_5 = E_9 - E_{10} \quad D_6 = \mathcal{E} - E_1 - E_6 - E_{11} \quad D_7 = E_{12} - E_{13} \quad D_8 = E_{13} - E_{14}$

$D_9 = E_{14} - E_{15} \quad D_{10} = E_1 - E_2 - E_3 \quad D_{11} = E_6 - E_7 - E_8 \quad D_{12} = E_{11} - E_{12} - E_{13}.$

We denote the sub-lattice of the Picard group $\sum_{i=0}^{12} \mathbb{Z}D_i$ as $\langle D_i \rangle$. Let $\langle \alpha_i \rangle$ be the orthogonal (with respect to the intersection form) complement of $\langle D_i \rangle$ and let $\{\alpha_1, \alpha_2, \alpha_3\}$ be its basis. Notice that φ preserves $\langle D_i \rangle$ and $\langle \alpha_i \rangle$ because its action on the Picard group is a Cremona isometry (the action preserves the intersection numbers).

The mapping (11) is an expression of the action φ on $\mathrm{Pic}(X)$ by the basis $\{\mathcal{E}, E_1, \dots, E_{15}\}$, but $\{D_0, D_1, \dots, D_{12}, \alpha_1, \alpha_2, \alpha_3\}$ is a better basis for calculation of the degree of φ_n (we may consider $\mathrm{Pic}(X)$ to be a vector space on $\mathbb{C}$ instead of a $\mathbb{Z}$-module for this purpose). The reason is that $\langle D_i \rangle$ and $\langle \alpha_i \rangle$ are eigenspaces of φ and independent of each other, and moreover the action of φ on $\langle D_i \rangle$ is just a permutation. Hence it is enough to investigate the action on $\langle \alpha_i \rangle$ in order to know the level of growth of $\deg(\varphi_n)$.

Actually, by taking a basis of $\langle \alpha_i \rangle$ as

$$\alpha_1 = 2\mathcal{E} - 2E_1 - E_2 - E_3 - E_4 - E_5$$
$$\alpha_2 = 2\mathcal{E} - 2E_6 - E_7 - E_8 - E_9 - E_{10}$$
$$\alpha_3 = 2\mathcal{E} - 2E_{11} - E_{12} - E_{13} - E_{14} - E_{15}$$

(the fact is, this basis is the basis of the root system by regarding the $\langle \alpha_i \rangle$ and intersection form as the root lattice and the bilinear form respectively) and writing an element of $\langle \alpha_i \rangle$ as $r_1\alpha_1 + r_2\alpha_2 + r_3\alpha_3$, the action of φ on $\langle \alpha_i \rangle$ is expressed as

$$\begin{pmatrix} \overline{r_1} \\ \overline{r_2} \\ \overline{r_2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 2 & 2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_2 \end{pmatrix} \tag{13}$$

and the absolute values of its eigenvalues are $1, (3 \pm \sqrt{5})/2$.

Remark. The non-autonomous version is as follows:

$$\varphi : (x, y; a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto \left(y, -x + y + a_5 + \frac{a_1}{y^2} + \frac{a_2}{a_1 y}; a_6, -a_7 + 2a_5^2 a_6 \right.$$
$$\left. + \frac{2a_2a_6}{a_1}, a_1, a_2, -a_5, a_3, a_4 + 2a_3a_5^2 - \frac{2a_2a_3}{a_1} \right) \tag{14}$$

where $a_i \in \mathbb{C}$ and a_1, a_3, a_6 are nonzero [6]. In this case the coefficients of $\mathcal{E}$ and E_i do not change and therefore the degrees and the algebraic entropy do not change, since its action on the Picard group is identical to the action of the original autonomous version.

3.2. *The second discrete Painlevé equation*

We consider the second discrete Painlevé equation [10], $\mathrm{dP_{II}}$:

$$\varphi : (x, y; a_1, a_2, a_0) \mapsto (\bar{x}, \bar{y}; \bar{a_1}, \bar{a_2}, \bar{a_0})$$
$$= \left(-x - y + \frac{a_0}{y} + s, -y - \bar{x} + \frac{a_2 - \lambda}{\bar{x}} + s : a_1, a_2 - \lambda, a_0 + \lambda \right)$$

where $\lambda = a_1 + a_2 + a_0$. The mapping φ is written as $\varphi = \psi^2$, where ψ is defined as

$$\psi : (x, y; a_1, a_2, a_0) \mapsto (\bar{x}, \bar{y}; \bar{a_1}, \bar{a_2}, \bar{a_0})$$
$$= \left(y, -x - y + \frac{a_0}{y} + s : a_0 + a_2, -a_0, a_0 + a_1 \right).$$

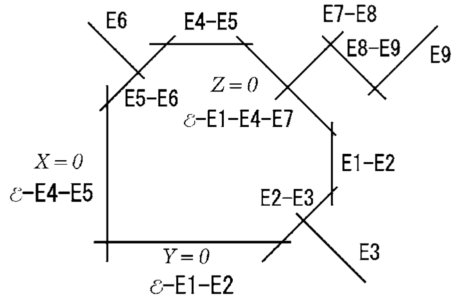

Figure 2. Proper transformation of $\mathrm{dP_{II}}$.

We define the corresponding surfaces X_i $(= X_{a_1,a_2,a_0})$ obtained by nine successive blow-ups from $\mathbb{P}^2$ as follows (figure 2):

$$\begin{aligned}
&E_1 : (Y/X, Z/X) = (0, 0)\\
&E_2 : (u_1, v_1) := (Z/X, Y/Z) = (0, 0)\\
&E_3 : (u_2, v_2) := (u_1, v_1/u_1) = (0, a_0)\\
&E_4 : (Z/Y, X/Y) = (0, 0)\\
&E_5 : (u_4, v_4) := (Z/Y, X/Z) = (0, 0)\\
&E_6 : (u_5, v_5) := (u_4, v_4/u_4) = (0, -a_2)\\
&E_7 : (Z/X, Y/X) = (0, -1)\\
&E_8 : (u_7, v_7) := (Z/X, (X + Y)/Z) = (0, s)\\
&E_9 : (u_8, v_8) := (u_7, (v_7 - s)/u_7) = (0, a_1).
\end{aligned}$$

The action of ψ, $(= (\psi^{-1})^*)$, on the Picard group is

$$\psi : \begin{pmatrix} \mathcal{E}, \ E_1 \\ E_2, \ E_3, \\ E_4, \ E_5, \ E_6, \ E_7, \ E_8, \ E_9 \end{pmatrix} \mapsto \begin{pmatrix} 2\mathcal{E} - E_4 - E_5 - E_6, \ \mathcal{E} - E_5 - E_6 \\ \mathcal{E} - E_4 - E_6, \ \mathcal{E} - E_4 - E_5 \\ E_7, \ E_8, \ E_9, \ E_1, \ E_2, \ E_3 \end{pmatrix}. \tag{15}$$

The Jordan normal form of ψ consists of the cells

$$\left\{ 1, 1, -1, \mathrm{e}^{\pi\sqrt{-1}/6}, \mathrm{e}^{\pi\sqrt{-1}/6}, \mathrm{e}^{\pi\sqrt{-1}/3}, \mathrm{e}^{\pi\sqrt{-1}/3}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Hence the order of the degree of the nth iterate is n^2.

On the other hand the anti-canonical divisor $-K_X$ can be reduced to prime divisors as

$$D_1 + 2D_2 + 3D_3 + 2D_4 + D_5 + 2D_6 + D_0$$

where

$$\begin{aligned}
&D_1 = E_2 - E_3 \quad D_2 = E_1 - E_2 \quad D_3 = \mathcal{E} - E_1 - E_4 - E_5\\
&D_4 = E_4 - E_5 \quad D_5 = E_5 - E_6 \quad D_6 = E_7 - E_8 \quad D_0 = E_8 - E_9.
\end{aligned}$$

The action of ψ on $\langle D_i \rangle$ is also just a permutation. The basis of the orthogonal complement of $\langle D_i \rangle$ is

$$\begin{aligned}
&\alpha_1 = \mathcal{E} - E_7 - E_8 - E_9\\
&\alpha_2 = \mathcal{E} - E_1 - E_2 - E_3\\
&\alpha_0 = \mathcal{E} - E_4 - E_5 - E_6.
\end{aligned}$$

The action of ψ on $\langle\alpha_i\rangle$ is

$$(\alpha_1, \alpha_2, \alpha_0) \mapsto (\alpha_0 + \alpha_2, -\alpha_0, \alpha_0 + \alpha_1).$$

The Jordan normal form consists of the cells

$$\left\{-1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\}.$$

Hence the coefficient of α_i of the nth iterate is of the order of n. Therefore the order of the degree of ψ cannot be observed only by the action on the root lattice in this case. The reason is that $\langle\alpha_i\rangle$ and $\langle D_i\rangle$ are not independent. This is the point of the proof in the next section.

4. The growth of degree of discrete Painlevé equations

In this section we prove that the degree of the nth iterate for every discrete Painlevé equation is $\mathrm{O}(n^2)$. The proof is based on the construction and the corresponding root systems.

It is shown by Sakai [10] that the discrete Painlevé equations can be obtained by the following method.

Let X be a rational surface obtained by blow-ups from $\mathbb{P}^2$ such that its anti-canonical divisor $-K_X\ (= 3\mathcal{E} - E_1 - \cdots - E_9)$ is uniquely decomposed in prime divisors as $-K_X = \sum_{i=1}^{I} m_i D_i$ and satisfies $K_X \cdot D_i = 0$ for all i. This implies that $K_X \cdot K_X = 0$ and therefore X is obtained by nine blow-up points from $\mathbb{P}^2$ and hence $\mathrm{rankPic(X)} = 10$. One can classify such surfaces according to the type (denoted by R) of Dynkin diagram formed by the D_i (the lattice of R is a sub-lattice of the lattice of $E_8^{(1)}$).

The Cremona isometries of X preserve the sub-lattice $\langle D_i\rangle$ and its orthogonal sub-lattice with respect to the intersection form. (Notice that the Cremona isometries correspond not only to the automorphisms of X but also the isomorphisms of X, where X is considered to be a surface parametrized by the points blown up. This fact corresponds to the fact that the Painlevé equations are a nonautonomous mapping.) By taking a suitable basis of the orthogonal lattice, $\{\alpha_1, \alpha_2, \ldots, \alpha_J\}$, and by regarding $\langle\alpha_j\rangle$ and the intersection form as the root lattice and the bilinear form respectively, it becomes the basis of an extended affine Weyl group and moreover $\alpha_j \cdot \alpha_j$ does not depend on j. All the actions of these extended affine Weyl groups on $\langle\alpha_j\rangle$ are uniquely extended to the actions on $\mathrm{Pic}(X)$ as Cremona isometries. Notice that the intersection number of α_j and K_X is zero, since $\alpha_j \cdot K_X = \alpha_j \cdot \sum m_i D_i = 0$. Similar to the case of the HV equation, every discrete Painlevé equation acts on $\{D_i\}$ just as a permutation (this fact follows from the uniqueness of decomposition of the anti-canonical divisor and the definition of Cremona isometry).

The group of Cremona isometries of X is isomorphic to the extended affine Weyl group and each element can be realized as a Cremona transformation, i.e. birational mapping, on $\mathbb{P}^2$. Each of the discrete Painlevé equations corresponds to a translation of the extended affine Weyl group.

The Cartan matrixes of these affine Weyl groups are symmetric and $-K_X$ becomes the canonical central element (and also becomes δ, see sections 6.2 and 6.4 in [17]). Hence the action of the Painlevé equation on the orthogonal lattice $\langle\alpha_j\rangle$ is expressed as

$$(\alpha_1, \alpha_2, \ldots, \alpha_J) \mapsto (\alpha_1 + k_1 K_X, \alpha_2 + k_2 K_X, \ldots, \alpha_J + k_J K_X) \tag{16}$$

where $k_j \in \mathbb{Z}$ and $\sum k_j = 0$.

Lemma 4.1. *Let X, D_i and α_j be as mentioned above. The formula*

$$\mathrm{rank}\langle D_1, \ldots, D_I, \alpha_1, \ldots, \alpha_J\rangle = 9$$

holds with respect to the rank.

Proof. Notice that $\{D_1, \ldots, D_I\}$ or $\{\alpha_1, \ldots, \alpha_J\}$ are linearly independent. Suppose $\sum d_i D_i + \sum r_j \alpha_j = 0$, where $d_i, r_j \in \mathbb{C}$. We have $F := -\sum d_i D_i = \sum r_j \alpha_j \in \langle D_i \rangle \cap \langle \alpha_j \rangle$. Since F is an element of $\langle D_i \rangle$, $\alpha_i \cdot (\sum r_j \alpha_j) = 0$ holds for all $1 \leqslant i \leqslant J$. Here the Cartan matrix of the Weyl group is $C := (c_{i,j})_{1 \leqslant i,j \leqslant J}$:

$$c_{i,j} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_i}$$

and $\alpha_i \cdot \alpha_i$ does not depend on i. Hence it implies

$$C\boldsymbol{r} = 0 \tag{17}$$

where $\boldsymbol{r} = (r_1, \ldots, r_J)$. The corank of a Cartan matrix of affine type is 1. Hence we obtain $F \in \mathbb{Z}K_X$. This implies the fact that the corank of $\langle D_1, \ldots, D_I, \alpha_1, \ldots, \alpha_J \rangle$ is 1. □

Let E_9 be an exceptional curve, where '9' means the last blow-up.

Lemma 4.2.

$$\{D_1, \ldots, D_I, \alpha_1, \ldots, \alpha_J, K_X, E_9\}$$

is a basis of $\mathrm{Pic}(X)$.

Of course these elements are not independent.

Proof. Suppose $E_9 = \sum d_i D_i + \sum r_j \alpha_j$, where $d_i, r_j \in \mathbb{C}$. Multiplying this equation by K_X, we find $-1 = 0$. The claim of the lemma follows from lemma 4.1. □

Let T be a discrete Painlevé equation. Since T acts on $\{D_i\}$ just as a permutation, there exists l such that T^l acts on $\{D_i\}$ as the identity.

Lemma 4.3. *There exist integers* $z_1, z_2, \ldots, z_J$ *such that*

$$T^l(E_9) = E_9 + \sum z_j \alpha_j$$

holds.

Proof. Notice that E_9 has an intersection with only one of $\{D_i\}$ and without loss of generality we can assume $E_9 \cdot D_1 = 1$. The system of equations $T^l(E_9) \cdot D_1 = 1$, $T^l(E_9) \cdot D_i = 0$ $(i = 2, \ldots, I)$ is linear. Hence the solutions of this system are $T^l(E_9) = E_9 + \sum \mathbb{C}\alpha_j$. Of course $T^l(E_9)$ must be an element of $\mathrm{Pic}(X)$ and therefore the coefficients must be integers. □

By (16), lemmas 4.2 and 4.3 the action of T^l on $\mathrm{Pic}(X)$ is expressed as

$$\begin{aligned} &d_1 D_1 + \cdots + d_I D_I + r_1\alpha_1 + \cdots + r_J\alpha_J + kK_X + eE_9 \\ &\qquad \mapsto d_1 D_1 + \cdots + d_I D_I + (r_1 + ez_1)\alpha_1 + \cdots + (r_J + ez_J)\alpha_J \\ &\qquad +(k + lr_1k_1 + \cdots + lr_Jk_J)K_X + eE_9 \end{aligned}$$

where $d_i, r_j, k, e \in \mathbb{Z}$. This action is written by the matrix

$$A := \left[\begin{array}{ccc|ccc|c|c} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ \hline & & & 1 & & & & z_1 \\ & & & & \ddots & & & \vdots \\ & & & & & 1 & & z_J \\ \hline & & & lk_1 & \cdots & lk_J & 1 & \\ \hline & & & & & & & 1 \end{array}\right] \tag{18}$$

where a blank means 0.

The matrix A^s, where $s \in \mathbb{N}$, is

$$A^s := \left[\begin{array}{ccc|ccc|c|c} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ \hline & & & 1 & & & & sz_1 \\ & & & & \ddots & & & \vdots \\ & & & & & 1 & & sz_J \\ \hline & & & slk_1 & \cdots & slk_J & 1 & *_s \\ \hline & & & & & & & 1 \end{array}\right] \tag{19}$$

where $*_s = \frac{1}{2}s(s-1)\sum lk_j z_j$.

Let us start with $\mathcal{E} \in \mathrm{Pic}(X)$ and let $d_1D_1 + \cdots + d_ID_I + r_1\alpha_1 + \cdots + r_J\alpha_J + kK_X + eE_9$ be an expression of $\mathcal{E}$. We obtain the following theorem.

Theorem 4.4. *For all discrete Painlevé equations the order of degree of the nth iterate is (at most)* $\mathrm{O}(n^2)$.

Proof. The degree of the Painlevé equation T as a birational mapping of $\mathbb{P}^2$ coincides with the coefficient of $\mathcal{E}$ in $T^n(\mathcal{E})$ as an action on $\mathrm{Pic}(X)$. Because the coefficients of

$$T^{sl}(\mathcal{E}) = \sum_i d_iD_i + \sum_j (r_j + sz_je)\alpha_j + \left(sl\sum_j k_jr_j + k + \tfrac{1}{2}s(s-1)le\sum_j k_jz_j\right)K_X + eE_9 \tag{20}$$

where $n = sl$, increase at most with the order s^2, the coefficient of $\mathcal{E}$ also increases at most with $\mathrm{O}(n^2)$. □

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